

COUNTING PATTERNS IN COLORED ORTHOGONAL ARRAYS

A. MONTEJANO AND O. SERRA

ABSTRACT. Let S be an orthogonal array $OA(d, k)$ and let c be an r -coloring of its ground set X . We give a combinatorial identity which relates the number of vectors in S with given color patterns under c with the cardinalities of the color classes. Several applications of the identity are considered. Among them, we show that every equitable r -coloring of the integer interval $[1, n]$ has at least $\frac{1}{2}(\frac{n}{r})^2 + O(n)$ monochromatic Schur triples. We also show that in an orthogonal array $OA(d, d-1)$, the number of monochromatic vectors of each color depends only on the number of vectors which miss that color and the cardinality of the color class.

1. INTRODUCTION

Arithmetic Ramsey Theory can be seen as the study of the existence of monochromatic structures, like arithmetic progressions or solutions of linear systems, in every coloring of sets of integers. The early results in the area are the theorem of Schur on monochromatic solutions of the equation $x + y = z$, the Van der Waerden theorem on monochromatic arithmetic progressions or the common generalization of the two, Rado's theorem, on monochromatic solutions of linear systems. Anti-Ramsey results refer to the study of combinatorial structures with elements of pairwise distinct colors, or rainbow structures, a subject started by Erdős, Simonovits and Sós [6] which has received much attention since then. Canonical Ramsey theory collects results ensuring the existence of either a monochromatic or a rainbow structure. Jungić, Licht, Mahdian, Nešetřil and Radoičić [9] started what they call Rainbow Ramsey Theory, which concerns the study of rainbow structures in colourings of sets of integers.

Counting versions of Arithmetic Ramsey results have also been obtained. Frankl, Graham and Rödl [7] prove that, in a finite coloring of an integer interval, actually a positive fraction of all solutions of a partition regular system are monochromatic. They also prove that, for every coloring of an integer interval, a positive fraction of all solutions of such a system are either monochromatic or rainbow.

Some of the above phenomena on the existence and number of color patterns in combinatorial structures behave in a particularly nice way when considered in finite groups. A simple example is the fact that the total number of monochromatic Schur triples in every two-coloring of the group of integers modulo n depends only on the cardinality of the color classes but not on the distribution of the colors, a fact first noticed, as far as we know, by Datskowsky [5]. The same is true for monochromatic three-term arithmetic progressions when n is relatively prime with 6, as noted by

Croot [4]. In [3] a combinatorial counting argument was given which explains the above two results and provides the ground for further generalizations in three directions. First, results like the above mentioned ones can be extended to general finite groups. Actually the universe to be colored needs not to be even a group, but simply the base set of an orthogonal array. Second, the monochromatic structures include Schur triples, arithmetic progressions, or solutions of more general equations in groups. Third, the counting argument can be applied to colorings with more than two colors and can also be used to study rainbow structures or specific color patterns. Of course there are limitations in such general results, which become less precise with the increasing complexity of the structures we consider.

We give in Section 2 a general formulation of the basic counting lemma (Lemma 1) which is based in a counting argument used in [3]. Section 3 collects some specific applications for orthogonal arrays $OA(3, 2)$, which include solutions of linear equations $ax + by + cz = d$ in an abelian group of order coprime with a , b and c or, more generally, equations of the form $x^\alpha y^\beta z^\gamma = b$ in a group G , where α, β, γ are automorphisms of G . In this case Lemma 1 leads to a relationship between monochromatic and rainbow triples which depends only on the cardinalities of the color classes (Theorem 2). This relationship provides results on the minimum number of monochromatic or of rainbow triples in orthogonal arrays (Corollary 3 and Corollary 6).

In the general context of orthogonal arrays $OA(3, 2)$ the relationship between monochromatic and rainbow triples can not be strengthened as illustrated in Example 4. Section 4 particularizes to linear equations of the form $ax + by + cz = d$ in an abelian group. In particular Theorem is used to obtain a lower bound on the number of monochromatic Schur triples in an equitable r -coloring of the integer interval $[1, n]$ (Theorem 8). For 3-term arithmetic progressions, colorings which are rainbow-free have been characterized in [11]. This characterization shows that every coloring with smaller color class sufficiently large has a rainbow triple. Here we obtain a similar result for general groups non necessarily abelian (Corollary 9). It is also shown that every equitable coloring of an abelian group has at least a linear number of rainbow solutions of any given linear equation which is partition regular (Corollary 10).

Section 5 is devoted to orthogonal arrays of the form $OA(d, d - 1)$. The main result establishes a relationship between monochromatic vectors of a given color and vectors which miss that color, which again depends only on the cardinality of the color class (Theorem 11). Particular instances are the main result in [3] on monochromatic vectors in 2-colorings, and a result by Balandraud [2] on rainbow vectors in 3-colorings.

The paper closes by considering the general case of orthogonal arrays $OA(d, k)$ in Section 6 where Lemma 1 is used to show that, for every r -coloring of the base set of an orthogonal array $OA(d, k)$, a positive proportion of all vectors has patterns in every ball of radius $(d - k)$, where we use the ℓ_1 distance in the set of r -vectors identifying color patterns (Theorem 13). This result provides a quantitative estimation which is particularized in the case of almost monochromatic (all but at

most one entry of the same color) or almost rainbow (all but at most one of the entries pairwise distinct) patterns (Theorem ??).

2. A COUNTING ARGUMENT

Let X be a finite set with cardinality n and let S be a set of vectors in X^d . Let $c : X \rightarrow [1, r]$ be an r -coloring of X with color classes X_1, \dots, X_r . A vector $\mathbf{x} = (x_1, x_2, \dots, x_d) \in S$ is *monochromatic* under c if all its coordinates belong to the same color class. When there are either no two coordinates of the same color class or all colors are present, we say that the vector is *rainbow* under c . We denote by $M = M(S)$ and $R = R(S)$ the set of monochromatic and rainbow vectors in S respectively.

A set S of d -vectors with entries in X is an *orthogonal array* of degree d and strength k if, for any choice of k columns, each k -vector of X^k appears in exactly one vector of S . In other words, if we specify any set of k entries a_1, \dots, a_k and any set of subscripts $1 \leq i_1 < i_2 < \dots < i_k \leq d$, we find exactly one vector $\mathbf{y} = (y_1, y_2, \dots, y_d)$ in S with $y_{i_1} = a_1, y_{i_2} = a_2, \dots, y_{i_k} = a_k$. We denote by $OA(d, k)$ the family of orthogonal arrays of degree d and strength k on X .

Lemma ?? below is the basic tool we shall use. It is based on the counting arguments used in [3].

In what follows we use the following notation. The color classes of an r -coloring of X will be denoted by X_1, X_2, \dots, X_r , and we denote by $c_i = |X_i|/n$ the density of the i -th color class. For a vector $\mathbf{u} = (u_1, \dots, u_r)$ with nonnegative integer entries, we denote by $|\mathbf{u}| = \sum_i u_i$. The multinomial coefficient $\binom{d}{u_1, u_2, \dots, u_r, d-|\mathbf{u}|}$ will be written as $\binom{d}{\mathbf{u}}$. For a vector $\mathbf{v} = (v_1, \dots, v_r)$ we write $\binom{\mathbf{v}}{\mathbf{u}} = \binom{v_1}{u_1} \binom{v_2}{u_2} \dots \binom{v_r}{u_r}$. We use the convention $\binom{v}{u} = 0$ if $v < u$ and $\binom{0}{0} = 1$.

Lemma 1. *Let S be an orthogonal array $OA(d, k)$ on X and let c be an r -coloring of X .*

For each vector $\mathbf{u} = (u_1, u_2, \dots, u_r)$ with $|\mathbf{u}| \leq k$ the following equality holds:

$$(1) \quad \frac{1}{n^k} \sum_{|\mathbf{v}|=d} \binom{\mathbf{v}}{\mathbf{u}} s(\mathbf{v}) = \binom{d}{\mathbf{u}} c_1^{u_1} \dots c_r^{u_r},$$

where the sum is extended to all vectors $\mathbf{v} = (v_1, v_2, \dots, v_r)$ with nonnegative integer entries, $|\mathbf{v}| = d$, and $s(\mathbf{v})$ is the number of vectors in S with v_i coordinates in X_i for each $i = 1, \dots, r$.

Proof. Given an ordered partition $V = (V_1, V_2, \dots, V_r)$ of $[1, d]$, possibly with some empty parts, let us denote by $S(V)$ the set of vectors in S whose entries in V_i belong to X_i , $1 \leq i \leq r$. An r -tuple of subsets (U_1, U_2, \dots, U_r) of $[1, d]$ is of type $\mathbf{u} = (u_1, u_2, \dots, u_r)$ if $|U_i| = u_i$, $1 \leq i \leq r$. Denote by $\mathcal{P}^r(\mathbf{u})$ the set of all r -tuples of pairwise disjoint subsets of $[1, d]$ of type \mathbf{u} . We say that V dominates U , and write $V \succeq U$, if $V_i \supset U_i$, $1 \leq i \leq r$.

Since S is an orthogonal array $OA(d, k)$, there are $k - |\mathbf{u}|$ vectors in S which meet a prescribed assignment of $|\mathbf{u}|$ coordinates. Hence, for each r -tuple of subsets (U_1, U_2, \dots, U_r) in $\mathcal{P}^r(\mathbf{u})$ there are $|X_1|^{u_1} |X_2|^{u_2} \dots |X_r|^{u_r} |X|^{k-|\mathbf{u}|}$ vectors in S whose entries in U_i belong to X_i , $1 \leq i \leq r$. Among these vectors we find all vectors in $S(V)$ for each partition V which dominates U , that is,

$$\sum_{V \succeq U} S(V) = |X_1|^{u_1} |X_2|^{u_2} \dots |X_r|^{u_r} |X|^{k-|\mathbf{u}|}.$$

Each partition V dominates $\binom{|V_1|}{u_1} \binom{|V_2|}{u_2} \dots \binom{|V_r|}{u_r}$ r -tuples in $\mathcal{P}^r(\mathbf{u})$. Summing up through all r -tuples in $\mathcal{P}^r(\mathbf{u})$ we get vectors counted by $s(\mathbf{v})$ for each \mathbf{v} which dominates componentwise the vector \mathbf{u} :

$$\begin{aligned} \binom{d}{\mathbf{u}} |X_1|^{u_1} |X_2|^{u_2} \dots |X_r|^{u_r} |X|^{k-|\mathbf{u}|} &= \sum_{U \in \mathcal{P}^r(\mathbf{u})} \sum_{V \succeq U} S(V) \\ &= \sum_{V \succeq \mathbf{u}} \sum_{U \in \mathcal{P}^r(\mathbf{u})} S(V) \\ &= \sum_{|\mathbf{v}|=d} \binom{v_1}{u_1} \binom{v_2}{u_2} \dots \binom{v_r}{u_r} \sum_{V \in \mathcal{P}^r(\mathbf{v})} S(V) \\ &= \sum_{|\mathbf{v}|=d} \binom{v_1}{u_1} \binom{v_2}{u_2} \dots \binom{v_r}{u_r} s(\mathbf{v}). \end{aligned}$$

Dividing by n^k we get equation (1). \square

Lemma 1 gives a relationship between the number of vectors with some specific color patterns and the cardinalities of the color classes. This identity may provide some precise formulas for the number of vectors with a particular color pattern, or at least approximate counting results of a general nature. In the remaining of the paper we give some applications of these identities.

3. COLOUR PATTERNS IN $OA(3, 2)$

For orthogonal arrays $O(3, 2)$ we get a nice relationship between monochromatic and rainbow vectors.

Theorem 2. *Let S be an orthogonal array $OA(3, 2)$ on X and $n = |X|$. For any r -coloring of X we have*

$$(2) \quad 2|M| - |R| = n^2 \left(3 \sum_{i=1}^r c_i^2 - 1 \right),$$

where M and R denote the set of monochromatic and rainbow vectors of S respectively.

Proof. By taking $\mathbf{u} = (0, 0, 0)$ in Lemma 1 we get

$$(3) \quad |X|^2 = \sum_{\|v\|=3} s(\mathbf{v}) = |M| + |R| + |T(2, 1)|,$$

where $T(2, 1) = S \setminus \{M \cup R\}$ denotes the set of vectors in S with exactly two entries of the same colour.

On the other hand, the choice of $\mathbf{u} = (2, 0, 0)$ in Lemma 1 gives

$$3|X_1|^2 = 3s(3, 0, 0) + s(2, 1, 0) + s(2, 0, 1).$$

Adding up similar countings with $(0, 2, 0)$ and $(0, 0, 2)$, we have

$$(4) \quad 3 \sum_{i=1}^r |X_i|^2 = 3|M| + |T(2, 1)|.$$

The result follows by subtracting (3) from (4). \square

As an immediate consequence of Theorem 2 we get:

Corollary 3. *Let c be an r -coloring of the base set of an orthogonal array $OA(3, 2)$ with $\alpha_c = 3 \sum_{i=1}^r c_i^2 - 1$. If $\alpha_c > 0$ then there are at least $\alpha_c n^2$ monochromatic triples and, if $\alpha_c < 0$, then there are at least $|\alpha_c| n^2$ rainbow triples.*

In particular, every equitable coloring with $r \geq 4$ colors has at least

$$(1 - 3/r)n^2$$

rainbow triples.

In the context of orthogonal arrays there are examples which show that essentially all solutions for $|M|$ and $|R|$ in equation (2) are possible values for the number of monochromatic and rainbow vectors in an orthogonal array whose points are colored. We illustrate this fact with the following example.

Example 4. Let Y be a multiplicative quasigroup (we only require the cancellation law) and consider the quasigroup $X = Y \times \mathbb{Z}_3$ with $(x, i) * (y, j) = (xy, 2(i + j))$. The set of triples $\{(x, i), (y, j), (x, i) * (y, j) : (x, i), (y, j) \in X\}$ is an orthogonal array. The coloring $\chi(x, i) = i$ on X has the maximum possible number $3|Y|^2$ of monochromatic triples and the maximum possible number $6|Y|^2$ of rainbow triples for an equitable coloring of X .

Let L be the latin square on X with entries $L((x, i), (y, j)) = (x, i) * (y, j)$ for each $(x, i), (y, j) \in X$. Let $U, V \subset Y$ be subsets of Y . Exchange the entries in L of the form $(xy, 0)$ with $(xy, 1)$ for every $x \in U$ and $y \in V$. The resulting orthogonal array has $3|Y|^2 - 2|U| \cdot |V|$ monochromatic triples for the same coloring of X .

Let L' be the latin square obtained by the above procedure with $U = V = Y$. For each pair $U', V' \subset Y$ we can now exchange the entries $(xy, 0)$ with $(xy, 2)$ whenever $x \in U'$ and $y \in V'$. The resulting orthogonal array has $|Y|^2 - |U'| \cdot |V'|$ monochromatic triples for the same coloring. By choosing $U' = V' = Y$, there are no monochromatic, and therefore no rainbow, triples. These are examples of equitable colorings in orthogonal arrays for each value of $|M| \in [0, |Y|^2] \cup (|Y|^2 + 2 \cdot [0, |Y|^2])$. \square

For two-colorings there are no rainbow triples, so that Theorem 2 gives a formula for the total number of monochromatic triples in terms of the cardinalities of the

color classes. By minimizing that formula (with each color class of density $1/2$) we get the minimum number of monochromatic triples in an orthogonal array $OA(3, 2)$ for any two-coloring of its ground set X . More precisely, we have the next Corollary which is a natural generalization of Corollary 3.1 in [3],

Corollary 5. *Let S be an orthogonal array $OA(3, 2)$ on X . For any 2-coloring of X we have*

$$|M| = |X_1|^2 - |X_1| \cdot |X_2| + |X_2|^2.$$

In particular, for any 2-coloring of X , there are at least $n^2/4$ monochromatic triples in S .

In the case of three-colorings Theorem 2 has a nice interpretation. Let us call

$$\sigma_c^2 = \sum_{i=1}^r c_i^2/r - \left(\sum_{i=1}^r c_i/r\right)^2$$

the *variance* of an r -coloring c . For $r = 3$ the expression on the right of equation (2) coincides, up to a constant, with the variance of the coloring (in particular is always nonnegative.) Theorem 2 can be restated for three-colorings in the following form.

Corollary 6. *Let S be an orthogonal array $OA(3, 2)$ on X . For any 3-coloring of X we have*

$$(5) \quad 2|M| - |R| = 9\sigma_c^2 n^2.$$

In particular, there are at least $(9\sigma_c^2/2)n^2$ monochromatic triples.

4. LINEAR EQUATIONS

Natural extensions of results in Arithmetic Ramsey Theory concern the study of color patterns of structures in groups. The results in Section 2 can be directly applied to this setting. The set of solutions of a linear equation of the form

$$ax + by + cz = d$$

in an abelian group of order coprime with a, b and c forms an orthogonal array $OA(3, 2)$. In this case more precise information on the number of monochromatic ir rainbow triples can be obtained, usually depending on the particular equation we are considering.

There are colorings with only rainbow triples. Take for instance the set of solutions of the equation

$$x + y + z = -1$$

in a cyclic group of order $n \equiv 0 \pmod{3t}$ for some $t \geq 1$. Consider the partition $A_i = [0, (n/3t) - 1] + i(n/3t)$, $0 \leq i \leq 3t - 1$. We have $A_i + A_i = [0, 2(n/3t) - 2] + 2i(n/3t)$ and $-A_i - 1 = [(3t - 1)n/(3t), n - 1] - i(n/3t)$, which are disjoint for each i , so that there are no monochromatic triples for that equation. Thus the lower bound for rainbow triples given in Corollary 3 is also best possible. Moreover, the same example for $t = 1$ shows that, for $\alpha_c = 0$, there are colorings of orthogonal arrays which have no monochromatic, and hence no rainbow, triples.

The lower bound on the number of monochromatic triples in Corollary 6 is also best possible. Consider for example Schur triples in a group G , triples of the form (x, y, z) with $xy = z$. The set of Schur triples in a finite group forms an orthogonal array $OA(3, 2)$. Alekseev and Sachev [1] proved that every equinumerous 3-coloring of the integers in $[1, 3n]$ contains a rainbow Schur-triple. The result was later improved by Schönheim [15] who proved that any 3-coloring of the integers in $[1, N]$ such that the smallest color class has more than $N/4$ elements contains a rainbow Schur triple, and this lower bound is best possible. The following example shows that for finite groups there are also 3-colorings with no rainbow Schur triples such that the smaller color class has cardinality $n/4$.

Example 7. Let $K < H < G$ be two subgroups of a finite group G such that K has index two in H and H has index two in G . Give color 1 to the elements in K , color 2 to the elements in $H \setminus K$ and color by 3 the remaining elements of the group. In this example $X_1X_2 = X_2$ and $X_1X_3 = X_2X_3 = X_3$. Thus there are no rainbow Schur triples under this coloring. \square

It is not clear to us that the lower bound $n/4$ for the size of the smaller color class is tight in the case of three-colorings of groups with no rainbow Schur triples.

Theorem 2 can also be used to estimate the minimum number of monochromatic triples in colorings of the integers. Robertson and Zeilberger [14] showed that the minimum number of monochromatic Schur triples in a two coloring of the integer interval $[1, n]$ is $n^2/11 + O(n)$. These authors exhibit a coloring with color classes of density $6/11$ and $5/11$ which attains the lower bound. The same result was obtained by Schoen [13] and Datskovsky [5]. The later author used Corollary 5 as an intermediate step of his proof. By using Theorem 2 one can obtain a simple proof of a lower bound on the number of Schur triples in an *equitable* coloring of $[1, n]$ with an arbitrary number of colors.

Theorem 8. *Any equitable r -coloring of the integer interval $[1, n]$ has at least*

$$|M| \geq (1/2r^2)n^2 + O(n)$$

monochromatic Schur triples.

Proof. Let $N = 2n$ and consider the $(r+1)$ coloring $\{X_1, X_2, \dots, X_r, X_{r+1}\}$ of the cyclic group $\mathbb{Z}/N\mathbb{Z}$ where $\{X_1, X_2, \dots, X_r\}$ is the given three-coloring of $[1, n]$ and $X_{r+1} = [n+1, 2n]$ (we identify the integers in $[1, 2n]$ with its representatives modulo N). We consider the Schur triples of $\mathbb{Z}/N\mathbb{Z}$ ordered as (x, y, z) with $x + y = z$.

By (2) we have

$$\begin{aligned} 2|M'| - |R'| &= (3 \sum_{i=1}^{r+1} c_i^2 - 1)(2n)^2 = (3(r(1/2r)^2 + (1/2)^2) - 1)4n^2 \\ (6) \qquad \qquad &= -(1 - 3/r)n^2, \end{aligned}$$

where M', R' are the sets of monochromatic and rainbow Schur triples respectively, and $c_i = |X_i|/2n$ are the densities of the color classes.

There are $|M_{X_{r+1}}| = n^2/2 + O(n)$ Schur triples of color X_{r+1} . The number $\sum_{i=1}^r |M_{X_i}|$ of monochromatic Schur triples of the other colors coincides, up to

$O(n)$ terms, with the number $|M|$ of monochromatic triples in the given coloring of the integer interval $[1, n]$.

Let us estimate the number $|R'|$ of rainbow triples. For each $u \in Y$, $Y \in \{X_1, \dots, X_r\}$, we have the triples

$$(u, w - u, w), \quad w \in [1, u] \setminus Y,$$

and the triples

$$(u - w, w, u), \quad w \in [u, n] \setminus Y.$$

Therefore, for each $u \in [1, n]$ there are $(1 - 1/r)n + O(1)$ such rainbow triples (each counted twice according to the permutation of the first two coordinates) giving rise to

$$(1 - 1/r)n^2 + O(n)$$

rainbow Schur triples with the third coordinate in $\cup_{i=1}^r X_i$. On the other hand, for each $u \in Y$, $Y \in \{X_1, \dots, X_r\}$, there also the rainbow Schur triples of the form

$$(n - u, w, n + w - u) \text{ and } (w, n - u, n + w - u), \quad w \in [u, n] \setminus Y$$

with the third coordinate in X_{r+1} . There are at least $(r - 1)n/r - u + O(1)$ choices for such w , giving a total of at least

$$\begin{aligned} 2 \sum_{u=1}^{(r-1)n/r} ((r - 1)n/r - u + O(1)) &= 2(1 - 1/r)^2 n^2 - (1 - 1/r)^2 n^2 + O(n) \\ &= (1 - 1/r)^2 n^2 + O(n) \end{aligned}$$

such rainbow triples. By plugging this estimation in (6) we get

$$|M| \geq \frac{1}{2} (((1 - 1/r) + (1 - 1/r)^2 - 1 - (1 - 3/r))n^2 + O(n)) = (1/2r^2)n^2 + O(n).$$

□

Let us consider next 3-term arithmetic progressions. Let G be a finite group and denote by $p(G)$ the smallest prime divisor of $|G|$. A d -term arithmetic progression in a finite group G with $p(G) \geq d$ is a set of the form $\{a, ax, ax^2, \dots, ax^{d-1}\}$ where $a, x \in G$. When G is abelian the set $AP(3)$ of 3-term arithmetic progressions correspond to solutions of the equation $x - 2y + z = 0$.

By proving a conjecture in [9] it was shown in [?] that a 3-coloring of an abelian group G of order n such that the smaller color class has cardinality at least $n/2p(G)$ does have rainbow $AP(3)$, and there are three-colorings of abelian groups in which the smallest color class has density $1/6$ and are free of rainbow $AP(3)$. For a non necessarily abelian group G the following can be proved.

Corollary 9. *A 3-coloring of a group G with $p(G) > 53$ with smaller color class of cardinality αn has at least $(6\alpha(2 - 3\alpha) - 29/15)n^2$ rainbow $AP(3)$. In particular, if $\alpha > (0.2725)n$ then there is a rainbow $AP(3)$.*

Proof. By Corollary 6, the number $|R|$ of rainbow $AP(3)$ satisfies

$$(7) \quad |R| = 2|M| - 9\sigma_c^2 n^2.$$

We have

$$9\sigma_c^2 = 3 \sum_{i=1}^3 c_i^2 - 1 \leq 3(2\alpha^2 + (1-2\alpha)^2) - 1 = 18\alpha^2 - 12\alpha + 2 = 6\alpha(2\alpha - 3) + 2.$$

For a group G with $p(G) > 53$, it is shown in [3] that every 3-coloring of G has at least $n^2/30$ monochromatic $AP(3)$. By substitution in (7) we get

$$|R| \geq (1/15 + 6\alpha(3 - 2\alpha) - 2)n^2 = (6\alpha(2 - 3\alpha) - 29/15)n^2.$$

The last part of the statement follows since the coefficient of n^2 in the above equation is positive if $\alpha > (0.2725)n$. \square

The equations $x+y-z=0$ and $x-2y+z=0$ for Schur triples and 3-term arithmetic progressions in abelian groups are examples of *regular* equations, namely, equations of the form $ax + by + cz = 0$ such that the sum of a nonempty subset of the coefficients is zero. As another consequence of Corollary 6 we have the following result concerning rainbow solutions of such equations.

Corollary 10. *Let $ax + by + cz = 0$ be a regular equation in an abelian group G of order n . For every equitable 3-coloring of G there are at least $2n$ rainbow solutions of the equation.*

Proof. If $a + b + c = 0$ then the system has the n solutions $\{(x, x, x) : x \in G\}$. If $a + b = 0$ then the system has the n solutions $\{(x, -x, 0) : x \in G\}$. For an equitable 3-coloring we have $\sigma_c^2 = 0$. Therefore Corollary 6 gives $|R| \geq 2|M| \geq 2n$. \square

5. COLOR PATTERNS IN $OA(d, d-1)$

For orthogonal arrays $OA(d, d-1)$ with arbitrary $d \geq 3$, Lemma 1 gives the following relation.

Theorem 11. *Let S be an orthogonal array $OA(d, d-1)$ on a set X with cardinality n . For each r -coloring of X and each color class X_i we have*

$$|S_i| + (-1)^{d-1}|M_i| = ((1 - c_i)^d - (-1)^d c_i^d) n^{d-1},$$

where S_i denotes the set of vectors in S which miss color i and M_i is the set of monochromatic vectors of color i . In particular, the total number $|M|$ of monochromatic vectors satisfies

$$(8) \quad \sum_{i=1}^r |S_i| + (-1)^{d-1}|M| = \sum_{i=1}^r ((1 - c_i)^d - (-1)^d c_i^d) n^{d-1}.$$

Proof. Without loss of generality we may assume $i = 1$. Consider the alternating sum of the equations (1) for vectors of type $\mathbf{u}_j = (j, 0, \dots, 0)$, $j = 0, 1, \dots, d-1$.

We have

$$\begin{aligned}
\sum_{j=0}^{d-1} (-1)^j \binom{d}{j} |X_1|^j |X|^{d-1-j} &= \sum_{j=0}^{d-1} (-1)^j \sum_{\|\mathbf{v}\|=d} \binom{v_1}{j} s(\mathbf{v}) \\
&= \sum_{\|\mathbf{v}\|=d} \left(\sum_{j=0}^{d-1} (-1)^j \binom{v_1}{j} \right) s(\mathbf{v}) \\
&= \sum_{\|\mathbf{v}\|=d, v_1=0} s(\mathbf{v}) + (-1)^{d+1} s(d, 0, \dots, 0) \\
(9) \qquad \qquad \qquad &= |S_1| + (-1)^{d+1} |M_1|,
\end{aligned}$$

where S_1 denotes the set of vectors which miss color 1 and M_1 denotes the set of vectors with all entries of color 1. The first term of the above equalities can be written as

$$\frac{1}{n} ((n - |X_1|)^d - (-1)^d |X_1|^d) = ((1 - c_1)^d - (-1)^d c_1^d) n^{d-1},$$

which gives the first part of the statement. Equality (8) is simply obtained by adding up the equations (9) for $i = 1, \dots, d$. \square

For 2-colorings, S_1 and S_2 are just the set of monochromatic vectors of color 2 and 1 respectively. Thus equation (8) shows that, for d odd, the number of monochromatic vectors depends only on the cardinalities of the color classes independently of their distribution, which is the main result in [3].

In particular we get the following Corollary for 3-colorings, which is a slight generalization of a result by Balandraud [2, Corollary 2]. Here a vector is said to be *rainbow* if all colors are present.

Corollary 12. *Let S be an orthogonal array $OA(d, d-1)$ on X . For each 3-coloring of X we have*

$$(1 + (-1)^{d-1})|M| - |R| = \left(\sum_{i=1}^3 ((1 - c_i)^d - (-1)^d c_i^d) - 1 \right) n^{d-1}.$$

Proof. With our current notion of rainbow vectors we have

$$|R| = |\cap_{j=1}^3 \bar{S}_j| = |S| - \sum_{i=1}^3 |S_i| + |M|.$$

By substitution in the last equation of Theorem 11 we have

$$\begin{aligned}
\sum_{i=1}^3 |S_i| + (-1)^{d+1} |M| &= |X|^{d-1} - |R| + (1 + (-1)^{d+1}) |M| \\
&= \sum_{i=1}^r ((1 - c_i)^d - (-1)^d c_i^d) n^{d-1},
\end{aligned}$$

as claimed. \square

It follows from Corollary 12 that, for d even, the number of rainbow vectors in a 3-coloring of the base set of an orthogonal array $OA(d, d-1)$ depends only on the cardinality of the color classes but not on the distribution of the colors.

6. COLOR PATTERNS IN $OA(d, k)$

Orthogonal arrays $OA(d, k)$ include sets of solutions of linear systems: for a $(d \times m)$ integer matrix A such that every $(m \times m)$ submatrix is nonsingular, the set of solutions of the linear system $Ax = b$ in an abelian group forms an orthogonal array $OA(d, d-m)$. In this general context Lemma 1 still provides some information on the distribution of color patterns. Recall that a color pattern in an r -coloring of the base set of an orthogonal array $OA(d, k)$ is identified by a vector $\mathbf{v} = (v_1, \dots, v_r)$ with $\sum_i v_i = d$ where entry v_i denotes the number of appearances of color i . We consider the distance between two color r -vectors \mathbf{u}, \mathbf{w} given by $d(\mathbf{u}, \mathbf{w}) = \sum_i |u_i - w_i|$.

Theorem 13. *Let S be an orthogonal array $OA(d, k)$ on a set X and let c be an r -coloring of X with $\alpha = \min_i c_i$. For each color pattern $\mathbf{v} = (v_1, \dots, v_r)$ there is a color pattern $\mathbf{v}' = (v'_1, \dots, v'_r)$ at distance $d(\mathbf{v}, \mathbf{v}') \leq 2(d-k)$ such that there are at least*

$$s(\mathbf{v}') \geq \frac{1}{\binom{d-k+r-1}{r-1}} (\alpha n)^k$$

vectors of S colored with \mathbf{v}' .

Proof. We say that an r -vector \mathbf{w} dominates the r -vector \mathbf{u} , written $\mathbf{w} \succeq \mathbf{u}$, if $w_1 \geq u_1, \dots, w_r \geq u_r$.

Let \mathbf{v} be a given color pattern and choose $\mathbf{u} = (u_1, \dots, u_r)$ with $|\mathbf{u}| = k$ such that $\mathbf{v} \succeq \mathbf{u}$. For every vector \mathbf{v}' with $|\mathbf{v}'| = d$ we have $\binom{\mathbf{v}'}{\mathbf{u}} \leq \binom{d}{\mathbf{u}}$. By equation (1),

$$\left(\binom{d}{\mathbf{u}} c_1^{u_1} \cdots c_r^{u_r} \right) n^k = \sum_{\mathbf{v}' \succeq \mathbf{u}} \binom{\mathbf{v}'}{\mathbf{u}} s(\mathbf{v}') \leq \binom{d}{\mathbf{u}} \sum_{\mathbf{v}' \succeq \mathbf{u}} s(\mathbf{v}').$$

There are at most $\binom{d-k+r-1}{r-1}$ vectors \mathbf{v}' with $|\mathbf{v}'| = d$ which dominate \mathbf{u} , and each such \mathbf{v}' is at distance $d(\mathbf{v}', \mathbf{v}) \leq d(\mathbf{v}', \mathbf{u}) + d(\mathbf{v}, \mathbf{u}) \leq 2(d-k)$ from \mathbf{v} . Hence, if $\alpha = \min\{c_1, \dots, c_r\}$,

$$\max_{\mathbf{v}' \succeq \mathbf{u}} s(\mathbf{v}') \geq \frac{1}{\binom{d-k+r-1}{r-1}} (c_1^{u_1} \cdots c_r^{u_r}) n^k \geq \frac{1}{\binom{d-k+r-1}{r-1}} (\alpha n)^k.$$

□

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UNIVERSITAT POLITÈCNICA DE CATALUNYA, JORDI GIRONA, 1, E-08034 BARCELONA, SPAIN, E-MAIL: OSERRA@MAT.UPC.ES